LESSON

11 Implicit Differentiation

A common complaint about the Chain Rule is "I don't know where to stop!" For example, why do we use the Chain Rule for $f(x) = \sin(x^3)$ to get $f'(x) = \cos(x^3) \cdot 3x^2$, but not for $g(x) = \sin(x)$, which has $g'(x) = \cos(x)$? The honest answer is that we *could* use the Chain Rule everywhere including in the following:

$$g'(x) = \cos(x) \cdot \frac{d}{dx}(x) = \cos(x) \cdot 1 = \cos(x)$$

$$f'(x) = \cos(x^3) \cdot \frac{d}{dx}(x^3) = \cos(x^3) \cdot 3x^2 \cdot \frac{d}{dx}(x) = \cos(x^3) \cdot 3x^2 \cdot 1 = \cos(x^3) \cdot 3x^2$$

When we get down to $\frac{d}{dx}(x) = 1$, we know we are done. The advantage to this way of thinking is that it explains what $\frac{dy}{dx}$ really means. This isn't merely a symbol that says "we took the derivative." This is the result of differentiating both sides of an equation.

Example

Differentiate $y = 4x^5 + e^x$.

Solution

Start with the equation.

$y = 4x^5 + e^x$

Differentiate both sides of the equation.

$\frac{d}{dx}(y) = \frac{d}{dx}(4x^5 + e^x)$

Use $\frac{d}{dx}(y) = \frac{dy}{dx}$.

$\frac{dy}{dx} = 20x^4 \cdot \frac{d}{dx}(x) + e^x \cdot \frac{d}{dx}(x)$

Simplify.

$\frac{dy}{dx} = 20x^4 \cdot 1 + e^x \cdot 1 = 20x^4 + e^x$

Now if $y = 4x^5 + e^x$, then there is a relationship between y and x. This relationship is given *explicitly* because we know exactly what y is in terms of x. However, if the variables x and y are all mixed up on both sides of the equals sign, then the relationship is given *implicitly*. The relationship is implied, but it is up to us to figure out what the relationship is explicitly. For example, the equation of the unit circle is:

$x^2 + y^2 = 1$

There is a relationship between the values of x and y, because what y can be depends on the value of x. If $x = 0$, for instance, then y could be either 1 or -1. We could take the implicit description of y in $x^2 + y^2 = 1$ and make it explicit by solving for y:

$y^2 = 1 - x^2$

$y = \pm\sqrt{1 - x^2}$

Solving for y is not always possible, though. If our equation were $\ln(y) + \cos(y) = 3e^x - x^3$, then we would not be able to solve for y.

Fortunately, we can still find the slope of the tangent line, $\frac{dy}{dx}$, without having to solve the original equation for y. The trick is to use *implicit differentiation* by taking the derivative of both sides and making sure to include $\frac{d}{dx}(y) = \frac{dy}{dx}$ wherever the Chain Rule dictates.

Example

Find the slope of the tangent line to $x^2 + y^2 = 1$.

Solution

Start with the equation.

$x^2 + y^2 = 1$

Differentiate both sides.

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1)$

Use the Chain Rule everywhere.

$2x \cdot \frac{d}{dx}(x) + 2y \cdot \frac{d}{dx}(y) = 0$

Use $\frac{d}{dx}(x) = 1$ and $\frac{d}{dx}(y) = \frac{dy}{dx}$.

$2x \cdot 1 + 2y \cdot \frac{dy}{dx} = 0$

Solve for $\frac{dy}{dx}$.

$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$

It might feel unpleasant to have $\frac{dy}{dx}$ given in terms of both x and y, but this is necessary. If we were

asked, "What is the slope of the tangent line to $x^2 + y^2 = 1$ at $x = \frac{1}{2}$?" We would have to reply, "Which one?" There are *two* tangent lines with $x = \frac{1}{2}$! See Figure 11.1. If we want the slope of the tangent line at $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, then

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

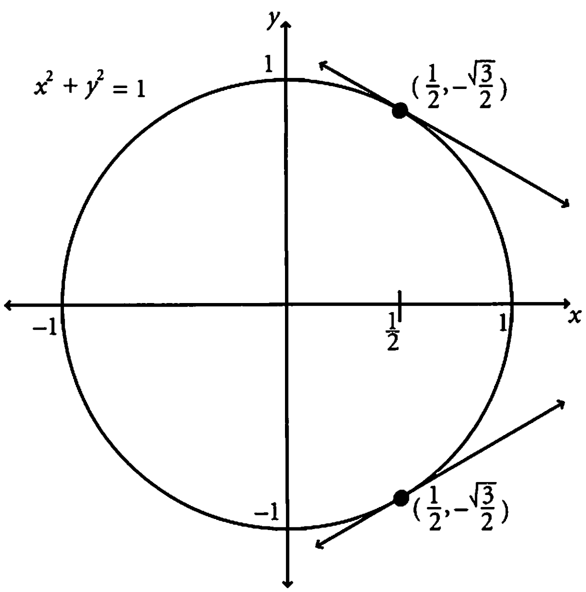

Figure 11.1

Example

Find $\frac{dy}{dx}$ when $\ln(y) + \cos(y) = 3e^x - x^3$.

Solution

Start with the equation.

$$\ln(y) + \cos(y) = 3e^x - x^3$$

Differentiate both sides of the equation.

$$\frac{d}{dx}(\ln(y) + \cos(y)) = \frac{d}{dx}(3e^x - x^3)$$

Use the Chain Rule everywhere.

$$\frac{1}{y}\cdot\frac{d}{dx}(y) - \sin(y)\cdot\frac{d}{dx}(y) = 3e^x\cdot\frac{d}{dx}(x) - 3x^2\cdot\frac{d}{dx}(x)$$

Use $\frac{d}{dx}(x) = 1$ and $\frac{d}{dx}(y) = \frac{dy}{dx}$.

$$\frac{1}{y}\cdot\frac{dy}{dx} - \sin(y)\cdot\frac{dy}{dx} = 3e^x - 3x^2$$

Factor out a $\frac{dy}{dx}$.

$$\left(\frac{1}{y} - \sin(y)\right)\frac{dy}{dx} = 3e^x - 3x^2$$

Solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{3e^x - 3x^2}{\frac{1}{y} - \sin(y)}$$

To get rid of the fraction-in-a-fraction, we can multiply the top and bottom by the denominator y that we want to eliminate:

$$\begin{aligned}\frac{dy}{dx} &= \frac{3e^x - 3x^2}{\frac{1}{y} - \sin(y)}\\ &= \left(\frac{3e^x - 3x^2}{\frac{1}{y} - \sin(y)}\right)\cdot\left(\frac{y}{y}\right)\\ &= \frac{3ye^x - 3x^2y}{1 - y\sin(y)}\end{aligned}$$

Example

Find the slope of the tangent line to $y^2\ln(x) = y + 5$ at $(1,-5)$.

Solution

Start with the equation.

$y^2\ln(x) = y + 5$

Differentiate both sides of the equation.

$\frac{d}{dx}(y^2\ln(x)) = \frac{d}{dx}(y + 5)$

Use the product rule on $y^2\ln(x)$.

$2y \cdot \frac{d}{dx}(y) \cdot \ln(x) + \frac{1}{x} \cdot \frac{d}{dx}(x) \cdot y^2 = \frac{d}{dx}(y) + 0$

Use $\frac{d}{dx}(x) = 1$ and $\frac{d}{dx}(y) = \frac{dy}{dx}$.

$2y \cdot \frac{dy}{dx} \cdot \ln(x) + \frac{1}{x} \cdot y^2 = \frac{dy}{dx}$

Plug in $x = 1$ and $y = -5$.

$2(-5) \cdot \frac{dy}{dx} \cdot \ln(1) + \frac{1}{1} \cdot (-5)^2 = \frac{dy}{dx}$

Use $\ln(1) = 0$.

$25 = \frac{dy}{dx}$

Thus, the slope of the tangent line at $(1,-5)$ is $\frac{dy}{dx} = 25$.

Example

Find $\frac{dy}{dx}$ when $\tan(y) = xy + 7$.

Solution

Start with the equation.

$\tan(y) = xy + 7$

Differentiate both sides of the equation.

$\frac{d}{dx}(\tan(y)) = \frac{d}{dx}(xy + 7)$

Use the product rule on xy.

$\sec^2(y) \cdot \frac{d}{dx}(y) = \frac{d}{dx}(x) \cdot y + \frac{d}{dx}(y) \cdot x + 0$

Use $\frac{d}{dx}(x) = 1$ and $\frac{d}{dx}(y) = \frac{dy}{dx}$.

$\sec^2(y) \cdot \frac{dy}{dx} = y + \frac{dy}{dx} \cdot x$

Bring both instances of $\frac{dy}{dx}$ to the same side.

$\sec^2(y) \cdot \frac{dy}{dx} - \frac{dy}{dx} \cdot x = y$

Factor out a $\frac{dy}{dx}$.

$(\sec^2(y) - x)\frac{dy}{dx} = y$

Solve for $\frac{dy}{dx}$.

$\frac{dy}{dx} = \frac{y}{\sec^2(y) - x}$

Example

Now we are able to use implicit differentiation and the fact that $\frac{d}{dx}(e^x) = e^x$ to prove that $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$.

Solution

If $y = \ln(x)$, then the derivative will be $\frac{dy}{dx}$.

$y = \ln(x)$

Raise both sides as powers of e.

$e^y = e^{\ln(x)}$

Since $\ln(x)$ and e^x are inverses, $e^{\ln(x)} = x$.

$e^y = x$

Differentiate both sides.

$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$

Use the Chain Rule.

$e^y \cdot \frac{dy}{dx} = 1$

Solve for $\frac{dy}{dx}$.

$\frac{dy}{dx} = \frac{1}{e^y}$

Use $e^y = e^{\ln(x)} = x$.

$\frac{dy}{dx} = \frac{1}{x}$

▶ Practice

Find $\frac{dy}{dx}$ in the following equations.

1. $(y + 1)^3 = x^4 - 8x$
2. $y^3 + y = \sin(x)$
3. $\sin(y) = 4x + 7$
4. $y - \sqrt{y} = \ln(x)$
5. $y^2 + x = 3x^4 + 8y$
6. $e^x + e^y = x^3$
7. $\tan(y) = \cos(x)$
8. $y = \sqrt{x + y}$
9. $\sin(x) - \sin(y) = x$
10. $y - \ln(y) = 10x^3 - 6x^2 + 4$
11. $(y + x^2)^4 = 10x$
12. $x^2y = y^4x^4$
13. $\frac{x}{y} + xy = x + y$
14. $\sec(y) + 9y = x^3\cos(y)$
15. Find the tangent line slope of $y^3 + x^2 = y^2 - 5y + 14$ at $(-3,1)$.
16. Find the tangent line slope of $x^3 + y^3 = 3y - x$ at $(1,-2)$.

17. Find the slope of the tangent line to $\ln(3y - 5) + x = y^2$ at (4,2).

18. Find the slope of the tangent line at (2,3) on the graph of $x^2y + y^2x = 30$.

19. Find the equation of the tangent line to $\sin(y) = x$ at the point $\left(\frac{1}{2}, \frac{\pi}{6}\right)$.

20. Find the equation of the tangent line to $x^2 + 6y = xy + 3$ at $(3, -2)$.